Problem 3.21

Test the energy-time uncertainty principle for the free particle wave packet in Problem 2.42 and the observable x, by calculating σ_H, σ_x, and $d\langle x\rangle/dt$ exactly.

Solution

The energy-time uncertainty principle states that

$$\Delta E\,\Delta t \geq \frac{\hbar}{2}.$$

Write ΔE and Δt in terms of σ_H, σ_x, and $d\langle x\rangle/dt$ with definitions.

$$\sigma_H \frac{\sigma_x}{\left|\frac{d\langle x\rangle}{dt}\right|} \geq \frac{\hbar}{2}$$

Use the definition of σ, the standard deviation.

$$\boxed{\sqrt{\langle H^2\rangle - \langle H\rangle^2}\,\frac{\sqrt{\langle x^2\rangle - \langle x\rangle^2}}{\left|\frac{d\langle x\rangle}{dt}\right|} \geq \frac{\hbar}{2}}$$

The aim now is to calculate all five quantities on the left side for a travelling gaussian free particle wave packet with position-space wave function,

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}} \exp\left[\frac{-a\left(x-\frac{\hbar lt}{m}\right)^2}{1+\frac{2i\hbar at}{m}}\right] \exp\left[il\left(x-\frac{\hbar lt}{2m}\right)\right].$$

Most of them were already calculated in Problem 2.42.

$$\langle x\rangle = \frac{\hbar lt}{m}$$

$$\langle x^2\rangle = \frac{m^2+4\hbar^2a^2t^2}{4m^2a} + \frac{\hbar^2l^2t^2}{m^2}$$

$$\frac{d\langle x\rangle}{dt} = \frac{\hbar l}{m}$$

$$\langle p\rangle = \hbar l$$

$$\langle p^2\rangle = \hbar^2(a+l^2)$$

The expectation values of H and H^2 at time t are as follows.

$$\begin{aligned}
\langle H \rangle = \langle \Psi \,|\, \hat{H} \,|\, \Psi \rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t) \hat{H} \Psi(x,t)\, dx \\
&= \int_{-\infty}^{\infty} \Psi^*(x,t) \left(\frac{\hat{p}^2}{2m} \right) \Psi(x,t)\, dx \\
&= \frac{1}{2m} \int_{-\infty}^{\infty} \Psi^*(x,t) \hat{p}^2 \Psi(x,t)\, dx \\
&= \frac{1}{2m} \int_{-\infty}^{\infty} \Psi^*(x,t) \left(-i\hbar \frac{\partial}{\partial x} \right)^2 \Psi(x,t)\, dx \\
&= -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \Psi^*(x,t) \frac{\partial^2 \Psi}{\partial x^2}\, dx
\end{aligned}$$

$$\begin{aligned}
\langle H^2 \rangle = \langle \Psi \,|\, \hat{H}^2 \,|\, \Psi \rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t) \hat{H}^2 \Psi(x,t)\, dx \\
&= \int_{-\infty}^{\infty} \Psi^*(x,t) \left(\frac{\hat{p}^2}{2m} \right)^2 \Psi(x,t)\, dx \\
&= \frac{1}{4m^2} \int_{-\infty}^{\infty} \Psi^*(x,t) \hat{p}^4 \Psi(x,t)\, dx \\
&= \frac{1}{4m^2} \int_{-\infty}^{\infty} \Psi^*(x,t) \left(-i\hbar \frac{\partial}{\partial x} \right)^4 \Psi(x,t)\, dx \\
&= \frac{\hbar^4}{4m^2} \int_{-\infty}^{\infty} \Psi^*(x,t) \frac{\partial^4 \Psi}{\partial x^4}\, dx
\end{aligned}$$

The problem with this formulation is that it's very tedious to compute the derivatives of $\Psi(x,t)$. It was done in Problem 2.42 to compute $\langle p \rangle$ and $\langle p^2 \rangle$; a linear function times an exponential had to be integrated in the former, and a quadratic function times an exponential had to be integrated in the latter. In order to compute $\langle H^2 \rangle$ this way, one would have to differentiate $\Psi(x,t)$ with respect to x four times, which would lead to a long quartic polynomial times an exponential function. To avoid this mess, one can instead write the expectation values of total energy in terms of the momentum-space wave function.

$$\begin{aligned}
\langle H \rangle = \langle \Phi \,|\, \hat{H} \,|\, \Phi \rangle &= \int_{-\infty}^{\infty} \Phi^*(p,t) \hat{H} \Phi(p,t)\, dp \\
&= \int_{-\infty}^{\infty} \Phi^*(p,t) \left(\frac{\hat{p}^2}{2m} \right) \Phi(p,t)\, dp \\
&= \frac{1}{2m} \int_{-\infty}^{\infty} \Phi^*(p,t) p^2 \Phi(p,t)\, dp \\
&= \frac{1}{2m} \langle \Phi \,|\, \hat{p}^2 \,|\, \Phi \rangle = \frac{1}{2m} \langle p^2 \rangle
\end{aligned}$$

$$\begin{aligned}
\langle H^2 \rangle = \langle \Phi \,|\, \hat{H}^2 \,|\, \Phi \rangle &= \int_{-\infty}^{\infty} \Phi^*(p,t) \hat{H}^2 \Phi(p,t)\, dp \\
&= \int_{-\infty}^{\infty} \Phi^*(p,t) \left(\frac{\hat{p}^2}{2m} \right)^2 \Phi(p,t)\, dp \\
&= \frac{1}{4m^2} \int_{-\infty}^{\infty} \Phi^*(p,t) p^4 \Phi(p,t)\, dp \\
&= \frac{1}{4m^2} \langle \Phi \,|\, \hat{p}^4 \,|\, \Phi \rangle = \frac{1}{4m^2} \langle p^4 \rangle
\end{aligned}$$

$\Phi(p,t)$ will not be any more complicated than $\Psi(x,t)$ because the Fourier transform of a gaussian is another gaussian.

Compute the momentum-space wave function by taking the Fourier transform of $\Psi(x,t)$ and completing the square in the exponent.

$$\begin{aligned}
\Phi(p,t) &= \mathscr{F}\{\Psi(x,t)\} \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t)\, dx \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}} \exp\left[\frac{-a\left(x-\frac{\hbar lt}{m}\right)^2}{1+\frac{2i\hbar at}{m}}\right] \exp\left[il\left(x-\frac{\hbar lt}{2m}\right)\right] dx \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}} \int_{-\infty}^{\infty} \exp\left[-\frac{ma}{m+2i\hbar at}\left(x-\frac{\hbar lt}{m}\right)^2 + \left(il-\frac{ip}{\hbar}\right)x - \frac{i\hbar l^2 t}{2m}\right] dx \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \frac{1/\sqrt{2\pi\hbar}}{\sqrt{1+\frac{2i\hbar at}{m}}} \int_{-\infty}^{\infty} \exp\left[-\frac{ma}{m+2i\hbar at}\left(x^2-\frac{2\hbar lt}{m}x+\frac{\hbar^2 l^2 t^2}{m^2}\right) + \left(il-\frac{ip}{\hbar}\right)x - \frac{i\hbar l^2 t}{2m}\right] dx \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \frac{1/\sqrt{2\pi\hbar}}{\sqrt{1+\frac{2i\hbar at}{m}}} \int_{-\infty}^{\infty} \exp\left[-\frac{ma}{m+2i\hbar at}x^2 + \left(il-\frac{ip}{\hbar}+\frac{2\hbar alt}{m+2i\hbar at}\right)x - \frac{i\hbar l^2 t}{2m} - \frac{\hbar^2 a l^2 t^2}{m(m+2i\hbar at)}\right] dx \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \frac{1/\sqrt{2\pi\hbar}}{\sqrt{1+\frac{2i\hbar at}{m}}} \int_{-\infty}^{\infty} \exp\left\{-\frac{ma}{m+2i\hbar at}\left[x^2+\frac{im(p-\hbar l)-2p\hbar at}{\hbar am}x+\frac{i\hbar l^2 t}{2ma}\right]\right\} dx \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \frac{1/\sqrt{2\pi\hbar}}{\sqrt{1+\frac{2i\hbar at}{m}}} \exp\left(\frac{\hbar l^2 t}{2im-4\hbar at}\right) \int_{-\infty}^{\infty} \exp\left\{-\frac{ma}{m+2i\hbar at}\left[x^2+\frac{im(p-\hbar l)-2p\hbar at}{\hbar am}x\right]\right\} dx \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \frac{1/\sqrt{2\pi\hbar}}{\sqrt{1+\frac{2i\hbar at}{m}}} \exp\left(\frac{\hbar l^2 t}{2im-4\hbar at}\right) \int_{-\infty}^{\infty} \exp\left\{-\frac{ma}{m+2i\hbar at}\left[x^2+\frac{im(p-\hbar l)-2p\hbar at}{\hbar am}x+\frac{[im(p-\hbar l)-2p\hbar at]^2}{4\hbar^2 a^2 m^2}\right]\right\} \\
&\qquad \times \exp\left\{\frac{ma}{m+2i\hbar at}\frac{[im(p-\hbar l)-2p\hbar at]^2}{4\hbar^2 a^2 m^2}\right\} dx
\end{aligned}$$

Combine the exponential functions without x and write the quantity in brackets as a square.

$$\Phi(p,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{1/\sqrt{2\pi\hbar}}{\sqrt{1+\frac{2i\hbar at}{m}}} \exp\left\{\frac{\hbar l^2 t}{2im - 4\hbar at} + \frac{ma}{m+2i\hbar at}\frac{[im(p-\hbar l) - 2p\hbar at]^2}{4\hbar^2 a^2 m^2}\right\}$$

$$\times \int_{-\infty}^{\infty} \exp\left\{-\frac{ma}{m+2i\hbar at}\left[x + \frac{im(p-\hbar l) - 2p\hbar at}{2\hbar am}\right]^2\right\} dx$$

Make the following substitution.

$$u = x + \frac{im(p-\hbar l) - 2p\hbar at}{2\hbar am}$$

$$du = dx$$

Consequently,

$$\begin{aligned}
\Phi(p,t) &= \left(\frac{2a}{\pi}\right)^{1/4} \frac{1/\sqrt{2\pi\hbar}}{\sqrt{1+\frac{2i\hbar at}{m}}} \exp\left(-\frac{m+2i\hbar at}{4\hbar^2 am}p^2 + \frac{l}{2\hbar a}p - \frac{l^2}{4a}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{a}{1+\frac{2i\hbar at}{m}}u^2\right) du \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{\sqrt{1+\frac{2i\hbar at}{m}}} \exp\left(-\frac{m+2i\hbar at}{4\hbar^2 am}p^2 + \frac{l}{2\hbar a}p - \frac{l^2}{4a}\right) \frac{\sqrt{\pi}}{\sqrt{a}}\sqrt{1+\frac{2i\hbar at}{m}} \\
&= \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{2\hbar a}} \exp\left(-\frac{m+2i\hbar at}{4\hbar^2 am}p^2 + \frac{l}{2\hbar a}p - \frac{l^2}{4a}\right) \\
&= \frac{1}{\sqrt[4]{2\pi\hbar^2 a}} \exp\left(-\frac{m+2i\hbar at}{4\hbar^2 am}p^2 + \frac{l}{2\hbar a}p - \frac{l^2}{4a}\right).
\end{aligned}$$

This momentum-space wave function can be used to calculate the expectation values of momentum more easily. Start with the expectation value of p at time t.

$$\begin{aligned}
\langle p \rangle &= \langle \Phi \,|\, \hat{p} \,|\, \Phi \rangle \\
&= \int_{-\infty}^{\infty} \Phi^*(p,t)(p)\Phi(p,t)\, dp \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt[4]{2\pi\hbar^2 a}} \exp\left(-\frac{m-2i\hbar at}{4\hbar^2 am}p^2 + \frac{l}{2\hbar a}p - \frac{l^2}{4a}\right)(p)\frac{1}{\sqrt[4]{2\pi\hbar^2 a}} \exp\left(-\frac{m+2i\hbar at}{4\hbar^2 am}p^2 + \frac{l}{2\hbar a}p - \frac{l^2}{4a}\right) dp \\
&= \frac{1}{\hbar\sqrt{2\pi a}} \int_{-\infty}^{\infty} p \exp\left(-\frac{1}{2\hbar^2 a}p^2 + \frac{l}{\hbar a}p - \frac{l^2}{2a}\right) dp \\
&= \frac{1}{\hbar\sqrt{2\pi a}} \int_{-\infty}^{\infty} p \exp\left[-\frac{1}{2\hbar^2 a}(p-\hbar l)^2\right] dp
\end{aligned}$$

Make the following substitution.

$$v = p - \hbar l \quad \rightarrow \quad p = v + \hbar l$$

$$dv = dp$$

Consequently,

$$\begin{aligned}
\langle p\rangle &= \frac{1}{\hbar\sqrt{2\pi a}}\int_{-\infty}^{\infty}(v+\hbar l)\exp\left(-\frac{1}{2\hbar^2 a}v^2\right)dv \\
&= \frac{1}{\hbar\sqrt{2\pi a}}\left[\underbrace{\int_{-\infty}^{\infty} v\exp\left(-\frac{1}{2\hbar^2 a}v^2\right)dv}_{=0} + \hbar l\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2\hbar^2 a}v^2\right)dv\right] \\
&= \frac{1}{\hbar\sqrt{2\pi a}}\left[\hbar l\cdot\frac{\sqrt{\pi}}{\sqrt{1}}\sqrt{2\hbar^2 a}\right] \\
&= \hbar l
\end{aligned}$$

as expected. Now calculate the expectation value of p^2 at time t.

$$\begin{aligned}
\langle p^2\rangle &= \langle\Phi\,|\,\hat{p}^2\,|\,\Phi\rangle \\
&= \int_{-\infty}^{\infty}\Phi^*(p,t)(p^2)\Phi(p,t)\,dp \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt[4]{2\pi\hbar^2 a}}\exp\left(-\frac{m-2i\hbar at}{4\hbar^2 am}p^2+\frac{l}{2\hbar a}p-\frac{l^2}{4a}\right)(p^2)\frac{1}{\sqrt[4]{2\pi\hbar^2 a}}\exp\left(-\frac{m+2i\hbar at}{4\hbar^2 am}p^2+\frac{l}{2\hbar a}p-\frac{l^2}{4a}\right)dp \\
&= \frac{1}{\hbar\sqrt{2\pi a}}\int_{-\infty}^{\infty}p^2\exp\left(-\frac{1}{2\hbar^2 a}p^2+\frac{l}{\hbar a}p-\frac{l^2}{2a}\right)dp \\
&= \frac{1}{\hbar\sqrt{2\pi a}}\int_{-\infty}^{\infty}p^2\exp\left[-\frac{1}{2\hbar^2 a}(p-\hbar l)^2\right]dp
\end{aligned}$$

Make the same v-substitution as before.

$$\begin{aligned}
\langle p^2\rangle &= \frac{1}{\hbar\sqrt{2\pi a}}\int_{-\infty}^{\infty}(v+\hbar l)^2\exp\left(-\frac{1}{2\hbar^2 a}v^2\right)dv \\
&= \frac{1}{\hbar\sqrt{2\pi a}}\int_{-\infty}^{\infty}(v^2+2\hbar lv+\hbar^2 l^2)\exp\left(-\frac{1}{2\hbar^2 a}v^2\right)dv \\
&= \frac{1}{\hbar\sqrt{2\pi a}}\left[\int_{-\infty}^{\infty}v^2\exp\left(-\frac{1}{2\hbar^2 a}v^2\right)dv+2\hbar l\underbrace{\int_{-\infty}^{\infty}v\exp\left(-\frac{1}{2\hbar^2 a}v^2\right)dv}_{=0}+\hbar^2 l^2\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2\hbar^2 a}v^2\right)dv\right] \\
&= \frac{1}{\hbar\sqrt{2\pi a}}\left[\frac{\sqrt{\pi}}{2}(2\hbar^2 a)^{3/2}+\hbar^2 l^2\cdot\frac{\sqrt{\pi}}{\sqrt{1}}\sqrt{2\hbar^2 a}\right] \\
&= \hbar^2(a+l^2)
\end{aligned}$$

Finally, calculate the expectation value of p^4 at time t.

$$\begin{aligned}
\langle p^4 \rangle &= \langle \Phi \,|\, \hat{p}^4 \,|\, \Phi \rangle \\
&= \int_{-\infty}^{\infty} \Phi^*(p,t)(p^4)\Phi(p,t)\,dp \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt[4]{2\pi\hbar^2 a}} \exp\left(-\frac{m - 2i\hbar a t}{4\hbar^2 a m}p^2 + \frac{l}{2\hbar a}p - \frac{l^2}{4a}\right)(p^4)\frac{1}{\sqrt[4]{2\pi\hbar^2 a}} \exp\left(-\frac{m + 2i\hbar a t}{4\hbar^2 a m}p^2 + \frac{l}{2\hbar a}p - \frac{l^2}{4a}\right) dp \\
&= \frac{1}{\hbar\sqrt{2\pi a}} \int_{-\infty}^{\infty} p^4 \exp\left(-\frac{1}{2\hbar^2 a}p^2 + \frac{l}{\hbar a}p - \frac{l^2}{2a}\right) dp \\
&= \frac{1}{\hbar\sqrt{2\pi a}} \int_{-\infty}^{\infty} p^4 \exp\left[-\frac{1}{2\hbar^2 a}(p - \hbar l)^2\right] dp
\end{aligned}$$

Make the same v-substitution as before.

$$\begin{aligned}
\langle p^4 \rangle &= \frac{1}{\hbar\sqrt{2\pi a}} \int_{-\infty}^{\infty} (v + \hbar l)^4 \exp\left(-\frac{1}{2\hbar^2 a}v^2\right) dv \\
&= \frac{1}{\hbar\sqrt{2\pi a}} \int_{-\infty}^{\infty} (v^4 + 4\hbar l v^3 + 6\hbar^2 l^2 v^2 + 4\hbar^3 l^3 v + \hbar^4 l^4) \exp\left(-\frac{1}{2\hbar^2 a}v^2\right) dv \\
&= \frac{1}{\hbar\sqrt{2\pi a}} \left[\int_{-\infty}^{\infty} v^4 \exp\left(-\frac{1}{2\hbar^2 a}v^2\right) dv + 4\hbar l \underbrace{\int_{-\infty}^{\infty} v^3 \exp\left(-\frac{1}{2\hbar^2 a}v^2\right) dv}_{=0}\right. \\
&\qquad + 6\hbar^2 l^2 \int_{-\infty}^{\infty} v^2 \exp\left(-\frac{1}{2\hbar^2 a}v^2\right) dv + 4\hbar^3 l^3 \underbrace{\int_{-\infty}^{\infty} v \exp\left(-\frac{1}{2\hbar^2 a}v^2\right) dv}_{=0} \\
&\qquad \left. + \hbar^4 l^4 \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\hbar^2 a}v^2\right) dv\right] \\
&= \frac{1}{\hbar\sqrt{2\pi a}} \left[\frac{3\sqrt{\pi}}{4}(2\hbar^2 a)^{5/2} + 6\hbar^2 l^2 \cdot \frac{\sqrt{\pi}}{2}(2\hbar^2 a)^{3/2} + \hbar^4 l^4 \cdot \frac{\sqrt{\pi}}{\sqrt{1}}\sqrt{2\hbar^2 a}\right] \\
&= \hbar^4(3a^2 + 6al^2 + l^4)
\end{aligned}$$

As a result,

$$\begin{aligned}
\langle H \rangle &= \frac{1}{2m}\langle p^2 \rangle = \frac{\hbar^2}{2m}(a + l^2) \\
\langle H^2 \rangle &= \frac{1}{4m^2}\langle p^4 \rangle = \frac{\hbar^4}{4m^2}(3a^2 + 6al^2 + l^4).
\end{aligned}$$

The left side of the boxed formula evaluates to

$$\sqrt{\langle H^2\rangle - \langle H\rangle^2}\,\frac{\sqrt{\langle x^2\rangle - \langle x\rangle^2}}{\left|\frac{d\langle x\rangle}{dt}\right|} = \sqrt{\left[\frac{\hbar^4}{4m^2}(3a^2+6al^2+l^4)\right] - \left[\frac{\hbar^2}{2m}(a+l^2)\right]^2}\,\frac{\sqrt{\left(\frac{m^2+4\hbar^2a^2t^2}{4m^2a} + \frac{\hbar^2l^2t^2}{m^2}\right) - \left(\frac{\hbar l t}{m}\right)^2}}{\left|\frac{\hbar l}{m}\right|}$$

$$= \sqrt{\frac{\hbar^4 a(a+2l^2)}{2m^2}}\,\frac{\sqrt{\frac{m^2+4\hbar^2a^2t^2}{4m^2a}}}{\frac{\hbar l}{m}}$$

$$= \frac{\hbar}{2}\sqrt{\left(1+\frac{a}{2l^2}\right)\left(1+\frac{4\hbar^2a^2t^2}{m^2}\right)}.$$

Since this square root yields a number greater than 1 for all time, the energy-time uncertainty principle holds.